

The Special Theory of Relativity

(3rd Lecture)

Relativistic Kinematics

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Recapitulation

- 1. The Special Theory of Relativity: Postulates of Special Relativity**
- 2. The measurements of various quantities may be different in different inertial frames.**
- 3. The laws of Physics would remain the same in all inertial frames.**
- 4. The laws of physics do not allow us to distinguish between different inertial frames.**
- 5. The principle of relativity does not claim that all inertial frames are same.**
- 6. The speed of light in free space is same in all inertial reference frames**

RELATIVISTIC KINEMATICS - 1

Course Outcome

After studying this unit you will be able to.....

- use the Lorentz transformation equations.
- Explain relativity of simultaneity.
- Deal with the relativistic phenomenon like length contraction and time dilation, twin paradox etc.
- Solve numerical problems based on Special Relativity.

Postulates of special relativity will be used to deduce the new coordinate transformation,



Lorentz transformation

The implications of these postulates for particle kinematics will also be discussed

The notion of absolute space and absolute time will be revised in a world view based on special theory of relativity

we shall revisit the concept of relativity or simultaneity. We will study how this concept raises serious doubts about the Newtonian assumptions that the measurements of distance between two points and the time intervals are the same for all observers.

This will become clear when you study the phenomena of **length contraction** and **time dilation**.

We will learn how the velocity of an object transforms in going from one inertial frame to another under the Lorentz coordinate transformation

Finally we will discuss the relativistic Doppler effect as an application of special relativity **to** Optics.

LORENTZ TRANSFORMATION

we have to find out a new transformation of space and time coordinates of a given event in the inertial frames S and S' moving with a velocity $\vec{v} = v\hat{i}$ relative to one another.

This new transformation should be consistent with the postulates of special relativity.

We will start by considering two inertial frames of reference, moving with velocity v relative to one another.

We will consider S to be the rest (laboratory) frame of reference and S' to be the moving frame which has velocity v in the positive x - direction of S .

Let both frames be rectangular with their axes parallel.

Finally, we define the origin of time $t = 0$ in S and $t'=0$ in S' when the two origins of S and S' coincide.

LORENTZ TRANSFORMATION

Before we determine this new coordinate transformation we should be clear about the method of assigning coordinates to an event in an inertial frame S .

We will assume that every observer is equipped with a standard clock and a standard of length e.g., a meter stick.

The observer can then assign Cartesian right-handed rectangular space coordinates (x, y, z) to any event in S .

Knowing the distance of the event and noting the time at which the observer receives the light signal from it, a time coordinate t can be assigned to the event. Such coordinates (x, y, z, t) are called **standard coordinates**.

LORENTZ TRANSFORMATION

Let us consider a light wave spreading out from a point source stationed at the origin of the frames at $t = t' = 0$.

The wave-front (i.e., the surface of equal phase) will be a sphere if observed in the reference frame S in which the source is at rest.

According to the postulates of special relativity the wave-front must also be a sphere when observed in the frame S' .

If the shape of the wave-front changes in S' then we can know that the source is moving.

This would violate the first postulate of special relativity theory which tells us that if S and S' are isolated then no experiment can help us know which one of these frames (S or S') is moving.

Hence we should not be able to tell from the shape of the wave-front whether the source is at rest or in uniform motion.

The shape of the wave-front as observed from S and S' has to be the same.

LORENTZ TRANSFORMATION

The equation of the spherical wave front in S emitted at the origin at $t = 0$ is

$$x^2 + y^2 + z^2 = c^2 t^2 \text{-----(1)}$$

The equation of the wave-front in S' must be

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \text{-----(2)}$$

where the speed of light c is the same in both S & S', according to the second postulate.

we have to look for another coordinate transformation consistent with the special theory of relativity.

We will impose a few conditions to make our task easier.

The new coordinate transformation should be consistent with the postulates of special relativity, i.e., we must have a transformation which satisfies the above two equations simultaneously.

LORENTZ TRANSFORMATION

We assume that space and time are homogeneous, i.e., all points in space and time are equivalent

Let us suppose that we measure a length or time interval of a specific event in a reference frame.

The results of the measurement should not depend on where or when the event occurs.

They should be the same at whichever point in space or time that event occurs. (This assumption simplifies our task considerably because this leads to a linear transformation.)

For example, suppose x' depends on the square of x , i.e., $x' = ax^2$

Then the length of a rod would be related as follows in the frames S and S' :

$$x'_2 - x'_1 = a(x_2^2 - x_1^2) \text{ -----(3)}$$

LORENTZ TRANSFORMATION

If $x_1 = 1$ and $x_2 = 2$, then $x'_2 - x'_1 = 3a$.

However, if $x_1 = 4$ and $x_2 = 5$, then $x'_2 - x'_1 = 9a$

Thus, for a quadratic or higher order transformation the measured length of rod in S' would depend on where it was situated in S .

This is inconsistent with the assumed homogeneity of space.

We assume the following relations

$$\left. \begin{aligned} x' &= a_1x + a_2t \\ y' &= y \\ z' &= z \\ t' &= b_1x + b_2t \end{aligned} \right\} \text{-----(4)}$$

consider a point for which $x' = 0$. In the S frame, it is moving along the positive x -axis with speed v .

$$\text{So its coordinate in } S \text{ is } x = vt \Rightarrow \text{for } x' = 0, \frac{dx}{dt} = v \text{-----(5)}$$

LORENTZ TRANSFORMATION

Similarly for a point at $x = 0$. In the S' frame, it is moving along the negative x' -axis with speed v .

So its coordinate in S' is $x' = -vt' \Rightarrow$ for $x = 0, \frac{dx'}{dt'} = -v$ -----(6)

Now if we put $x' = 0$ in the 1st equation of equation (4)

$$\begin{aligned}x' &= a_1x + a_2t \\ \Rightarrow a_1x + a_2t &= 0 \\ \Rightarrow \frac{dx}{dt} &= -\frac{a_2}{a_1} = v \text{ -----(7)}\end{aligned}$$

For $x = 0$ from the 1st equation of equation (4) $x' = a_2t$

And from the 4th equation of equation (4) $t' = b_1x + b_2t$

$$\Rightarrow t' = b_2t \Rightarrow t = \frac{t'}{b_2} \text{ -----(8)}$$

Hence

$$x' = a_2t = \frac{a_2}{b_2} t' \text{ -----(9)}$$

LORENTZ TRANSFORMATION

$$x' = a_2 t = \frac{a_2}{b_2} t' \Rightarrow \frac{dx'}{dt'} = \frac{a_2}{b_2} = -v \text{-----(10)}$$

So we can write $\frac{a_2}{b_2} = \frac{a_2}{a_1} \Rightarrow a_1 = b_2 \text{-----(11)}$

Let us now substitute the coordinate transformation given by Eqs. (4) and (11) into Eq. (2). The result is

$$\begin{aligned} & (a_1 x + a_2 t)^2 + y^2 + z^2 = c^2 (b_1 x + b_2 t)^2 \\ \Rightarrow & (a_1 x + a_2 t)^2 + y^2 + z^2 = c^2 (b_1 x + a_1 t)^2 \\ \Rightarrow & a_1^2 x^2 + a_2^2 t^2 + 2a_1 a_2 x t + y^2 + z^2 = c^2 b_1^2 x^2 + c^2 a_1^2 t^2 + 2c^2 a_1 b_1 x t \\ \Rightarrow & (a_1^2 - c^2 b_1^2) x^2 + y^2 + z^2 + (a_2^2 - c^2 a_1^2) t^2 + 2(a_1 a_2 - c^2 a_1 b_1) x t = 0 \text{-----(12)} \end{aligned}$$

Comparing eqn. (1) and eqn. (12) we can write

$$a_1^2 - c^2 b_1^2 = 1 \text{-----(13)}$$

$$a_2^2 - c^2 a_1^2 = -c^2 \text{-----(14)}$$

$$a_1 a_2 = c^2 a_1 b_1 \text{-----(15)}$$

LORENTZ TRANSFORMATION

The 1st equation of equation (4) is $x' = a_1x + a_2t \Rightarrow x' = a_1 \left(x + \frac{a_2}{a_1}t \right)$
 $\Rightarrow x' = a_1(x - vt)$ -----(15) [using equation(7)]

The 4th equation of equation (4) is $t' = b_1x + a_1t$ [here eqn. (11) has been used]

$$t' = a_1 \left(t + \frac{b_1}{a_1}x \right) \text{ now from equation (15) } b_1 = \frac{a_2}{c^2}$$

$$\Rightarrow t' = a_1 \left(t + \frac{a_2}{a_1 c^2}x \right)$$

$$\Rightarrow t' = a_1 \left(t - \frac{v}{c^2}x \right)$$
-----(16) [using equation(7)]

Now from equation (14) and (7) we can write

$$a_1 = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \text{-----(17)}$$

Hence from equation (15) $x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \text{-----(18)}$

and from equation (16) $t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} \text{-----(19)}$

LORENTZ TRANSFORMATION

Hence the Lorentz transformation equations are given by

$$\begin{aligned}x' &= \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \\y' &= y \\z' &= z \\t' &= \frac{t-\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}}\end{aligned}$$

Lorentz equations are linear in
x and t

Let us introduce two new parameters $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$

So we can write the Lorentz transformation equations as

$$\left. \begin{aligned}x' &= \gamma(x - vt) = \gamma(x - \beta ct) \\y' &= y \\z' &= z \\t' &= \gamma\left(t - \frac{v}{c^2}x\right) = \gamma\left(1 - \frac{\beta}{c}x\right)\end{aligned} \right\} \text{-----(20)}$$

LORENTZ TRANSFORMATION

From Lorentz transformation equations we can immediately see that v can never exceed c .

For $v > c$, the space and time coordinates become imaginary, which is a physical impossibility.

Thus, we arrived the following conclusion:

We cannot measure speeds greater than the speed of light; c is the limiting speed in the physical universe.

The inverse Lorentz transformation is obtained by

$$x \leftrightarrow x'; y \leftrightarrow y'; z \leftrightarrow z'; t \leftrightarrow t' \text{ and } v \leftrightarrow -v$$

And is given by

$$\left. \begin{aligned} x &= \gamma(x' + \beta ct') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{\beta}{c}x'\right) \end{aligned} \right\} \text{-----(21)}$$

Relativity of Simultaneity

Let us consider that two events are occurring simultaneously in the reference frame S . Let us assign the coordinates (x_1, y_1, z_1, t_1) and (x_2, y_2, z_2, t_2) to the events. In this case $x_1 \neq x_2$, and $t_1 = t_2$.

Using Lorentz transformation we now show that, in general, those events which are simultaneous in S are not simultaneous in another inertial frame S' moving uniformly relative to S at a speed v .

Let the coordinates of the two events in S' be (x'_1, y'_1, z'_1, t'_1) and (x'_2, y'_2, z'_2, t'_2) .

From the inverse Lorentz transformation [equation (21)] we can write

$$t_1 = \gamma \left(t'_1 + \frac{\beta}{c} x'_1 \right) \text{ and } t_2 = \gamma \left(t'_2 + \frac{\beta}{c} x'_2 \right)$$

$$\begin{aligned} \text{Now } t_1 = t_2 &\Rightarrow \left(t'_1 + \frac{\beta}{c} x'_1 \right) = \left(t'_2 + \frac{\beta}{c} x'_2 \right) \\ &\Rightarrow t'_1 = t'_2 + \frac{\beta}{c} (x'_2 - x'_1) \end{aligned}$$

Relativity of Simultaneity

Now from Lorentz Transformation we can write $x'_1 = \gamma(x_1 - \beta ct_1)$ and $x'_2 = \gamma(x_2 - \beta ct_2)$

Hence $x'_2 - x'_1 = \gamma(x_2 - x_1) - \gamma\beta c(t_2 - t_1) \Rightarrow x'_2 - x'_1 = \gamma(x_2 - x_1)$ as $t_2 = t_1$

So we can write $t'_1 = t'_2 + \frac{\beta}{c}(x'_2 - x'_1) = t'_2 + \frac{\gamma\beta}{c}(x_2 - x_1)$

As $x_1 \neq x_2$ hence $t'_1 \neq t'_2$

However, for $x_1 = x_2$ and $t_1 = t_2$, We have $t'_1 = t'_2$

Thus, if two events occur simultaneously at same position in one inertial frame, then they are simultaneous and occur at the same position in every other inertial frame of Reference.

Similarly, it can be shown that events occurring at the same point in space but at different times in S will appear to be occurring at different points in S' . Thus, if $x_1 = x_2$ and $t_1 \neq t_2$ in S then $x'_1 \neq x'_2$ in S'



Thank you